Numerical Dispersion and Stability Analysis of the FDTD Technique in Lossy Dielectrics

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***Abstract*—Two different extensions of the finite-difference time-domain (FDTD) method for the treatment of lossy dielectrics are considered: the time-average (TA) and the time-forward (TF) difference schemes. An analytical study of the stability properties and numerical dispersion of these schemes is presented. The stability analysis is based on the Von Neumann (Fourier series) method, while the numerical dispersion properties are established in terms of the numerical permittivity of discrete lossy dielectrics. The analytical stability limits are compared with those obtained numerically in previous works. The accuracy of the two schemes is compared by computing the reflection coefficient of a lossy dielectric slab.**

***Index Terms*—FDTD method, lossy dielectrics.**

I. Introduction

First proposed for isotropic lossless media, the finite-difference time-domain (FDTD) method was quickly extended to lossy dielectrics characterized by a static conductivity. To apply the FDTD technique to such materials, the Maxwell–Ampére equation is completed with a conduction current term. This term must then be suitability discretized. A straightforward discretization of this term is not useful because the electric field must be evaluated at the temporal instants that correspond to the magnetic field, and this discretization scheme is not properly staggered. To avoid this difficulty two different approaches have been proposed for discretizing the conduction current term: the first approach uses a time average [1], while the second approach uses forward differences in time [2]. We will refer to these approaches as the time-average (TA) and the time-forward (TF) schemes, respectively. Both techniques have coexisted during the last two decades, but until now no analytical study of their stability and numerical dispersion properties has been carried out. Recently, the numerical stability and accuracy properties of a third technique for treating lossy dielectrics, the exponential time-differencing method, have been published [3].

This letter presents an analytical study of the numerical stability conditions and numerical dispersion characteristics of the TA and TF schemes for the FDTD treatment of lossy dielectrics. The stability analysis is based on the Von Neumann method, and the numerical dispersion is analyzed in terms of the numerical permittivity. To compare the accuracy of the two schemes, we consider the computation of the reflection coefficient of a lossy dielectric slab.

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II. The FDTD Treatment of Lossy Dielectrics

For the sake of brevity and simplicity, instead of working directly with Maxwell's equations, we consider the wave equation for the electric field in a source-free, homogeneous lossy medium

$$\left(\frac{1}{\mu\epsilon}\sum_{\beta=x,y,z}\partial_\beta^2-\frac{1}{\tau}\partial_t-\partial_t^2\right)E_\alpha(\vec{r},t)=0$$

where $\tau=\frac{\epsilon}{\sigma}$ is the medium relaxation time. The application of the standard FDTD scheme leads to the following difference equations:

$$\left(\frac{1}{\mu\epsilon}\sum_{\beta=x,y,z}\frac{\delta_\beta^2}{\Delta_\beta^2}-\frac{\delta_t}{\tau\Delta_t}-\frac{\delta_t^2}{\Delta_t^2}\right)E_\alpha(\vec{r}_{E_\alpha},n\Delta_t)=0 \quad (1)$$

where δ_β and δ_t denote the central difference operator with respect to the spatial coordinate β and to the time, respectively. Analogously, Δ_β and Δ_t denote the spatial and temporal steps. The central difference operator with respect to time is defined as [4]

$$\delta_t f(n\Delta_t)=\delta_t f^n \equiv f^{n+\frac{1}{2}}-f^{n-\frac{1}{2}}$$

and thus

$$\delta_t^2 f^n\equiv\delta_t[\delta_t f^n]=f^{n+1}-2f^n+f^{n-1}.$$

This operator is defined analogously for the spatial coordinates. Note that the above dicretization is not useful because the lossy term must be evaluated at half time steps, where values are not available. Two different methods have been described to avoid this problem: the TA and the TF schemes. By using the TA scheme, the lossy term of (1) is discretized as

$$\frac{1}{\tau}\partial_t E_\alpha(\vec{r}_{E_\alpha},n\Delta_t)\approx\frac{\delta_t}{\tau\Delta_t}\left(\frac{E_\alpha^{n-\frac{1}{2}}(\vec{r}_{E_\alpha})+E_\alpha^{n+\frac{1}{2}}(\vec{r}_{E_\alpha})}{2}+O(\Delta_t^2)\right)$$

while in the TF scheme, the lossy term of (1) leads to

$$\frac{1}{\tau}\partial_t E_\alpha(\vec{r}_{E_\alpha},n\Delta t)\approx\frac{\delta_t}{\tau\Delta_t}\left(E_\alpha^{n+\frac{1}{2}}(\vec{r}_{E_\alpha})+O(\Delta_t)\right).$$

From the above expressions, it can be seen that the TA scheme preserves the second-order accuracy of the standard FDTD method, while the TF scheme has only first order accuracy in time.

III. Stability Analysis

To investigate the stability condition for the above schemes by means of the Von Neumann method, the corresponding difference equations are expressed in the transformed domain. This leads to a second-degree polynomial $P(Z)$ in the complex variable Z—often called the amplification factor. The condition for stability is that all the roots of $P(Z)$ must be inside, or on, the unit circle in the Z-plane, $|Z| \leq 1$. Further details on the foundations and applicability of the Von Neumann method can be found in [5].

A. The TA Scheme

For the TA scheme we obtain the following stability polynomial:

$$\left(1+\frac{\Delta_t}{2\tau}\right)Z^2+(4\nu^2-2)Z+\left(1-\frac{\Delta_t}{2\tau}\right)=0$$

where

$$\nu^2=\frac{\Delta_t^2}{\epsilon\mu}\sum_{\beta=x,y,z}\frac{\sin^2\theta_\beta}{\Delta_\beta^2}$$

and $\theta_\beta=\frac{k_\beta\Delta_\beta}{2}$, with k_β being the numerical wavenumber in the β direction. We find that, in order to fulfill $|Z|\leq 1$, the following stability conditions must be verified:

$$\Delta_t\leq\sqrt{\epsilon\mu}\left(\sum_{\beta=x,y,z}\frac{1}{\Delta_\beta^2}\right)^{-\frac{1}{2}}$$

with $\sigma\geq 0$. Therefore, the TA scheme has the same stability limit as the standard FDTD method for lossless media.

B. The TF Scheme

For the TF scheme we obtain the stability polynomial

$$\left(1+\frac{\Delta_t}{\tau}\right)Z^2+\left(4\nu^2-2-\frac{\Delta_t}{\tau}\right)Z+1=0$$

in this case, we find the following stability conditions:

$$\Delta_t\leq\sqrt{\epsilon\mu}\left(\sum_{\beta=x,y,z}\frac{1}{\Delta_\beta^2}\right)^{-\frac{1}{2}}(A+\sqrt{A^2+1}) \qquad (2)$$

and $\sigma\geq 0$, where

$$A=\frac{\sqrt{\epsilon\mu}}{4\tau}\left(\sum_{\beta=x,y,z}\frac{1}{\Delta_\beta^2}\right)^{-\frac{1}{2}}.$$

The factor $(A+\sqrt{A^2+1})$ is always greater than one. Therefore, the stability limit of the TF scheme is greater than the limit of the standard FDTD method for lossless media by this factor, whose value depends on the conductivity and the size of the discretization cell. However, the maximum value of Δ_t allowed by (2) is still smaller than that given by the Courant condition

$$\Delta_t\leq\frac{1}{v_{p_{\max}}}\left(\sum_{\beta=x,y,z}\frac{1}{\Delta_\beta^2}\right)^{-\frac{1}{2}}$$

where $v_{p_{\max}}$ is the maximum phase speed in the lossy dielectric.

IV. Numerical Dispersion Analysis

The numerical dispersion equation can be obtained by simply evaluating the stability polynomial on the unit circle of the Z-plane, that is, by letting $Z=\exp(j\omega t)$.

A. The TA Scheme

For this scheme the numerical dispersion equation reads

$$\hat{\epsilon}^{\mathrm{TA}}\mu\frac{\sin^2\theta_t}{\Delta_t^2}=\sum_{\beta=x,y,z}\frac{\sin^2\theta_\beta}{\Delta_\beta^2} \qquad (3)$$

where $\hat{\epsilon}^{\mathrm{TA}}$ is the numerical complex permittivity for the TA scheme, given by

$$\hat{\epsilon}^{\mathrm{TA}}=\epsilon-\frac{j\sigma\Delta_t}{2\tan\theta_t}$$

where $\theta_t=\frac{\omega\Delta_t}{2}$. It can be observed that the real part of the complex permittivity is not affected by the discretization process, and the imaginary part of $\hat{\epsilon}^{\mathrm{TA}}$ tends to the analytical value as the time step tends to zero.

B. The TF Scheme

The numerical dispersion equation obtained for this scheme has the same form as (3), but with a numerical complex permittivity given by

$$\hat{\epsilon}^{\mathrm{TF}}=\epsilon\left(1-\frac{\Delta_t}{2\tau}\right)-\frac{j\sigma\Delta_t}{2\tan\theta_t}.$$

In this case, the real part of $\hat{\epsilon}^{\mathrm{TF}}$ is affected by the term $\frac{\Delta_t}{2\tau}$, hence the relaxation time constant should be resolved well by the FDTD time step $(\Delta_t \ll \tau)$ to accurately model the permittivity of a lossy dielectric by means of the TF scheme.

V. Numerical Examples

The stability limit of the TF scheme has been studied previously—from a pure numerical viewpoint—for a one-dimensional (1-D) case [6]. In Table I, we compare the maximum time step attainable with the TF scheme $\Delta_{t_{\max}}^{f}$ calculated by using (2) with the results obtained previously [6, Table II]. For the sake of brevity, only the cases with $\epsilon=\epsilon_0$ and $\Delta_x=\frac{\lambda_c}{20}$ are shown, where λ_c is the wavelength in the lossy dielectric at the frequency $f=6$ kHz. Table I also includes the maximum time steps reachable by the TA scheme $\Delta_{t_{\max}}^{a}$ and by the Courant Condition $\Delta_{t_{\max}}^{C}$. It can be seen that the stability limits obtained numerically in [6, Table

TABLE I
COMPARISON OF THE MAXIMUM TIME STEP ATTAINABLE IN A LOSSY MEDIUM WITH $\epsilon = \epsilon_0$ AND $\Delta_x = \frac{\lambda_c}{20}$, BY MEANS OF: 1) THE TA SCHEME, $\Delta^a_{t_{\max}}$; 2) THE TF SCHEME CALCULATED BY (2), $\Delta^f_{t_{\max}}$; 3) THE TF SCHEME CALCULATED IN [6, TABLE II], $(\Delta^f_{t_{\max}})^*$; AND 4) THE THEORETICAL LIMIT ESTABLISHED BY THE COURANT CONTITION $\Delta^C_{t_{\max}}$

σ (S/m)	τ (μs)	$\Delta^a_{t_{\max}}/\tau$	$\Delta^f_{t_{\max}}/\tau$	$\left(\Delta^f_{t_{\max}}\right)^*/\tau$	$\Delta^C_{t_{\max}}/\tau$
10^{-6}	8.85	$.653$	$.768$	$.723$	$.941$
10^{-5}	$8.85 \cdot 10^{-1}$	2.39	4.22	4.07	9.41
10^{-4}	$8.85 \cdot 10^{-2}$	7.68	31.4	30.5	94.1
10^{-3}	$8.85 \cdot 10^{-3}$	24.3	298	294	941
10^{7}	$8.85 \cdot 10^{-13}$	$2.43 \cdot 10^{6}$	$2.96 \cdot 10^{12}$	$2.94 \cdot 10^{12}$	$9.41 \cdot 10^{12}$

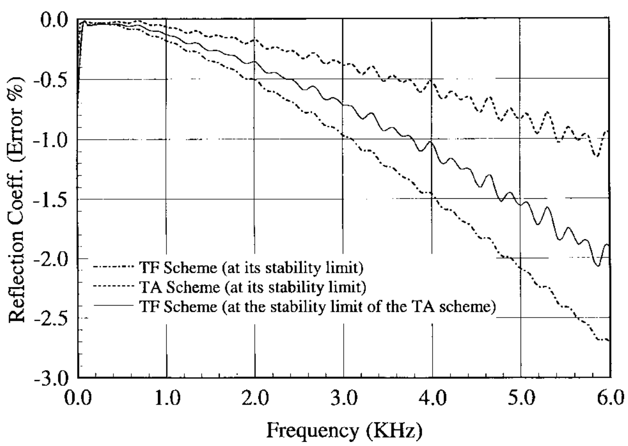

Fig. 1. Error in the magnitude of the reflection coefficient for a lossy dielectric slab, computed by using the TA and the TF schemes with different time steps.

II] are in good agreement with those established analytically in this work.

To compare the accuracy of the two schemes, Fig. 1 shows the relative error in the magnitude of the reflection coefficient for a lossy dielectric slab immersed in a lossless dielectric medium. The parameters taken for the lossy slab are $\epsilon = \epsilon_0$, $\sigma = 10^{-5}$ S/m, and its width is $d = 200\Delta_x$. In order to run the FDTD algorithm at the stability limit of the TF scheme without instabilities in the lossless dielectric, we have to take a dielectric constant for the lossless dielectric given by $\epsilon_r \gtrsim A + \sqrt{A^2+1}$, in this case we have taken $\epsilon_r = 3.2$. Fig. 1 shows three different curves: 1) the dashed line has been obtained with the TF scheme at its stability limit, i.e. by taking $\Delta_t = \Delta^f_{t_{\max}} \approx 3.73$ μs; 2) the dotted line has been computed with the TA scheme at its stability limit, i.e. by using $\Delta_t = \Delta^a_{t_{\max}} \approx 2.12$ μs; and 3) the solid line has been calculated with the TF scheme running at the stability limit of the TA scheme, i.e., by taking $\Delta_t = \Delta^a_{t_{\max}}$. It can be seen that, running at their respective limits, the TA scheme is much more accurate than the TF scheme. Furthermore, if the time step in the TF scheme is reduced to the limit of the TA scheme, the former gives results with a relative error that is approximately twice as large as the latter scheme.

VI. CONCLUSION

Analytical stability conditions for the TA and the TF schemes have been derived. Their numerical dispersion equations have also been established. The stability conditions show that, for a given spatial size of the discretization cell, the TF scheme allows higher values of Δ_t to be used than the TA scheme. However, the dispersion analysis shows that the TF scheme leads to a less accurate characterization of the permittivity of lossy dielectrics, mainly in situations where $\Delta_t \gtrsim \tau$. This is a direct consequence of the fact that the TF scheme does not preserve, for the truncation error, the second-order accuracy in time of the standard FDTD method. We have shown that running at their stability limits, the TA scheme is much more accurate than the TF scheme. Even if the time step for the TF scheme is reduced to the TA stability limit, the TA approach still gives better results. In conclusion, for the FDTD treatment of lossy dielectrics, the TA scheme is preferred over the TF scheme.

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